TITLE: STATIONARY AND TRAVELING LOADS IN A HOLLOW CYLINDER

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I.   INTRODUCTION

Just after World War II Dr. R. Beuwkes [1] of Watertown Arsenal
introduced the senior author to the theory of elastic stresses in thick-
walled cylinders and the technology of shell-pushing tests. The research
at Watertown Arsenal culminated in the publication of the Thick Walled
Cylinder Handbook [2], a monument to skill in analysis and computations on
a desk calculator. Investigation of non-metallic rotating bands at BRL led
to renewed interest in this area. We found that we were unable to inter-
polate in the tables cited above due to their limited accuracy; moreover
the value of Poisson's ratio used in the computations was not appropriate
for modern gun steels. An independent investigation was initiated, using
residue theory in place of Fourier series [3,4,5]. Since the eigenvalues
were complex, we required a subroutine for Bessel functions of integral
order and complex argument. The required subroutine was developed at BRL.
A Gauss continued fraction was used to reduce round off errors inherent in
series calculations [6,7]. A code giving accurate stresses on the outside
of the gun tube was developed [8,9]. This code was used to calculate
strains in a highly instrumented gun tube. The appropriate value of
Poisson's ratio and Young's modulus was obtained from the Benet Laboratory.
Agreement between theory and measurement was good at low velocities, but
systematic deviations were observed at high velocities. This result was
forecast in an early paper by G.S. Taylor [10], who used a dynamic version
of the Winkler theory for thin-walled tubes, but his results were
apparently ignored by the Army. A program based on scalar and vector wave
functions was initiated at BRL. The computations are difficult except for
torsion, which we discuss below. The theoretical work shows that the
equilibrium stress distribution is obtained when the velocity of travel
approaches zero in the limit, as one would expect on physical grounds.

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Recently we have resolved difficulties in calculating stresses near a discontinuity of loading on the inner surface of the cylinder. A method of calculating elastic stresses in thin-walled cylinders was also derived, so we are able to use the same mathematical formulation for wall ratios ranging from .01 to 5. Both of these problems required asymptotic methods and involved large values of the complex transform variable in the analysis.

II. NUMERICAL DIFFICULTIES

Formulation of boundary value problems for the infinite hollow cylinder has followed traditional lines and is not exceptionally difficult. Real problems arise in the numerical evaluation of Fourier integrals and the generation of Bessel functions of the second kind due to the integer arithmetic of the digital computer and its limited exponent range. Memory requirements and execution time are relatively modest for the class of problems under consideration. We have considered four types of error in the course of programming and numerical analysis.
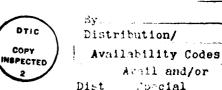
Round off error is persistent and insidious. It is very severe in the evaluation of Fourier integrals by quadratures along the real axis and was the principle reason why the calculus of residues was used in preference. It occurred in acute form in calculating Bessel functions of the second kind. This difficulty motivated our development of the subroutine cited above. In this paper we discuss round off error occurring in the manipulation of asymptotic series. Round off error is also a principle concern in generating special functions by recursion formulas, where it arises in connection with stability criteria.
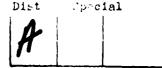
A continued fraction obviously can be used only for values of the variable and parameter for which division by zero will not occur. We finally are able to prove that division by zero would not occur in the portion of the subroutine using Gauss continued fractions. Theorems of Bucholz [11] and Hurwitz [12] were required in the proof [13]. The analysis is closely related to Hurwitz stability theory.

Serious truncation error has occurred only in evaluating residue series for the inner radius at points very close to the discontinuity of loading. Only recently have we found a method for improving the convergence of the residue series.

III. STRESSES NEAR A DISCONTINUITY OF LOADING FOR AXIALLY SYMMETIC STRESSES

For brevity we consider only axial stresses produced by a step function of pressure or shear applied to the inner cylindrical surface. The analysis of tangential stresses is similar. We superimpose a constant stress and a discontinuity stress to obtain the step function. For

pressure loading, we have

$$\tau_{rz} = 0, \quad \sigma_r = \tfrac{1}{2}\,\sigma_0, \quad r = a \quad \text{and} \tag{1a}$$

$$\tau_{rz} = 0, \quad \sigma_r = -\tfrac{1}{2}\,\sigma_0, \quad z<0; \quad \sigma_r = \tfrac{1}{2}\,\sigma_0, \quad z>0, \quad r = a, \tag{1b}$$

respectively, and for shear loading

$$\sigma_r = 0, \quad \sigma_{rz} = \tfrac{1}{2}\,\tau_0, \quad r = a \tag{2a}$$

$$\sigma_{rz} = 0, \quad \tau_{rz} = -\tfrac{1}{2}\tau_0, \quad z<0; \quad \tau_{rz} = \tfrac{1}{2}\tau0, \quad z>0, \quad r = a \tag{2b}$$

In both cases

$$\sigma_r = 0, \quad \tau_{rz} = 0, \quad r = b. \tag{3}$$

The solutions corresponding to (1a) and (2a) can be obtained by elementary methods and will not be considered here. The discontinuous stresses in (1b) and (2b) are represented by Cauchy discontinuous factors to facilitate solution by separation of variables.

The stresses are derived from Love's stress function [14] in the form

$$\phi_c = [AI_0(sr) + BK_0(sr) + CsrI_1(sr) + DsrK_1(sr)]\cos(sz) \tag{4}$$

for pressure loading and

$$\phi_s = [AI_0(sr) + BK_0(sr) + CsrI_1(sr) + DsrK_1(sr)]\sin(sz) \tag{5}$$

for shear loading. If the boundary conditions are homogeneous, we obtain four homogenous linear equations which are satisfied only if the determinant of the coefficients is equal to zero. In the case of shear loading, we obtain from Eq. (1) and a number of intermediate calculations the characteristic equation

$$\Delta_c = 0 \tag{6}$$

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where

$$\Delta_c(s) = \begin{vmatrix} I_0(p) & K_0(p) & \alpha_1 I_1(p) & \alpha_1 K_1(p) \\ I_1(p) & -K_1(p) & pI_0(p) & -pK_0(p) \\ I_0(q) & K_0(q) & \beta_1 I_1(q) & \beta_1 K_1(q) \\ I_1(q) & -K_1(q) & qI_0(q) & -qK_0(q) \end{vmatrix} \tag{7}$$

and

$$p = sa, \quad \alpha_1 = [\dot{p} + (2 - 2\nu)/p], \quad q = sb, \quad \beta_1 = [q + (2 - 2\nu)/q]. \tag{8}$$

The shear loading leads to the characteristic equation

$$\Delta_s(s) = 0 \tag{9}$$

where

$$\Delta_s(s) = -\Delta_c(s) \tag{10}$$

and obviously has the same characteristic roots.

The characteristic roots in the first quadrant of the complex s plane have the approximate value

$$s_n = t_n/(b-a) \tag{11}$$

where

$$t_n \sim \log_e[(2n-1)\pi] + i(n-\tfrac{1}{2})\pi, \quad n > 1 \tag{12}$$

The approximate values of s_n obtained from Eq. (11) are improved by Newton's method in the complex plane. It should be observed that all the determinants occurring in the analysis are analytic functions of s even though

logarithms occur in the series for the modified Bessel function of the second kind.

We find that

$$\sigma_z = \frac{\sigma_0}{\pi} \int_0^\infty (\Delta_1/\Delta_c)\sin(sz)ds/s \tag{13}$$

is the solution corresponding to pressure loading, Eq. (1b), and

$$\tau_{rz} = \frac{\tau_0}{\pi} \int_0^\infty (\Delta_2/\Delta_s)\cos(sz)ds/s \tag{14}$$

is the solution to the shear problem, Eq. (2b). The determinants Δ_1 and Δ_2 are given by

$$\Delta_1 = \begin{vmatrix} 0 & 0 & -2I_0(q) - \beta_2 I_1(q) & 2K_0(q) + \beta_2 K_1(q) \\ I_1(p) & -K_1(p) & pI_0(p) & -pK_0(p) \\ I_0(q) & K_0(q) & \beta_1 I_1(q) & \beta_1 K_1(q) \\ I_1(q) & -K_1(q) & qI_0(q) & -qK_0(q) \end{vmatrix} \tag{15}$$

$$\Delta_2 = \begin{vmatrix} I_0(p) & K_0(p) & 2I_0(p) + pI_1(p) & -2K_0(p) + pK_1(p) \\ I_1(p)/p & -K_1(p)/p & 3I_0(p) - \alpha_2 I_1(p) & -3K_0(p) - \alpha_2 K_1(p) \\ I_0(q) & K_0(q) & \beta_1 I_1(q) & \beta_1 K_1(q) \\ I_1(q) & -K_1(q) & qI_0(q) & -qK_0(q) \end{vmatrix} \tag{16}$$

where

$$\alpha_2 = (2-2\nu)/p, \qquad \beta_2 = (2-2\nu)/q \tag{17}$$

We obtain asymptotic approximations of the integrands by using Wronskian relations connecting $I_0(x)$, $I_1(x)$, $K_0(x)$, and $K_1(x)$, where $x = p$ or $x = q$, and the leading terms of the Hankel asymptotic expansions [15]. The leading terms are

$$I_0(x) \sim \frac{e^x}{(2\pi x)^{\frac{1}{2}}} \left[1 + \frac{1^2}{1!8x} + \frac{1^2 \cdot 3^2}{2!(8x)^2} \right] \tag{18}$$

$$I_1(x) \sim \frac{e^x}{(2\pi x)^{\frac{1}{2}}} \left[1 - \frac{1 \cdot 3}{1!8x} - \frac{1^2 \cdot 3 \cdot 5}{2!(8x)^2} \right] \tag{19}$$

$$K_0(x) \sim \frac{\pi}{2x}^{\frac{1}{2}} e^{-x} \left[1 - \frac{1^2}{1!8x} + \frac{1^2 \cdot 3^2}{2!(8x)^2} \right] \tag{20}$$

$$K_1(x) \sim \frac{\pi}{2x}^{\frac{1}{2}} e^{-x} \left[1 + \frac{1 \cdot 3}{1!8x} - \frac{1^2 \cdot 3 \cdot 5}{2!(8x)^2} \right] \tag{21}$$

We find

$$\Delta_1/(s\Delta_c) = s/(s+s_0), \qquad \Delta_2/(s\Delta_s) = -s/(s+s_0) \tag{22}$$

where

$$s_0 = (7-8\nu)(b-a)/4ab \tag{23}$$

On combining Eqs. (13), (14), and (22) we find the resulting integrals can be expressed in terms of sine and cosine integrals [16]. The approximation for small z follows from the fact that large values of the transform variable s correspond to small values of the argument z, according to the usual theory of Fourier integrals. Let γ be Euler's Constant in this context and let $z_0 = 1/s_0$ be a characteristic length. Then, when z is positive and very small, we have approximately

$$\sigma_z \simeq \frac{1}{2} \sigma_0 \tag{24}$$

for pressure loading and

$$\tau_{rz} \simeq \frac{\tau_0}{\pi} \left[\gamma + \log_e(z/z_0) \right] \tag{25}$$

for shear loading. We represent the logarithm by an integral of Fourier type [17, 18]. We subtract these dominant terms from the integrals given in Eqs. [13] and [14]. We obtain

$$\sigma_z = \sigma_0 \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Delta_1 - \Delta_c) \sin(sz)}{s \Delta_c(s)} \, ds \right] \tag{26}$$

$$\tau_{rz} = \frac{\tau_0}{\pi} \left[\log_e(z/z_0) + \int_0^\infty \frac{\Delta_2 \cos(sz_0) \, ds}{s \Delta_c(s)} \right. \tag{27}$$

$$\left. + \int_0^\infty \frac{(\frac{1}{2} \Delta_2 - \Delta_s)(2\cos(sz) - 2\cos(sz_0)) ds}{s \, \Delta_s(s)} \right]$$

The integrals in Eqs. (26) and (27) are more rapidly convergent than the original integrals in Eqs. (13) and (14) and will lead to more rapidly convergent residue series when the limits of integration are taken between $-\infty$ and ∞. The integrals must be re-written in exponential form as illustrated in the torsion problem to insure convergence of the contour integrals.

VI. ELASTIC STRESSES IN THIN WALLED CYLINDERS

We observe from Eqs. (11) and (12) that the eigenvalues of high order for a thin-walled cylinder become very large in absolute value. Exponential over-run then occurs when we use the Hankel asymptotic series to evaluate the various determinants. Moreover, when we use Laplace's reduction of the determinant in Eq. (7), we find expressions like $p[I_0^2(p) - I_1^2(p)$ and $p[K_0^2(p) - K_1^2(p)]$ occur, together with similar expressions involving q. When these expressions are evaluated by means of the Hankel asymptotic expansions, the leading terms are cancelled by subtraction, leading to increasingly severe round off error as the wall ratio approaches one. To overcome these difficulties, we obtained asymptotic expansions of these expressions in which the subtraction occurs algebraically rather than numerically. The exponentials were also combined algebraically, thus eliminating exponential over run for the range of wall ratios of interest.

Let [19]

$$w = Aw_1 + Bw_2 + Cw_3, \tag{28}$$

where

$$w_1 = \pi[pI_0{}^2(p) - pI_1{}^2(p)] \tag{29}$$

$$w_2 = [pI_0(p)K_0(p) + pI_1(p)K_1(p)] \tag{30}$$

$$w_3 = [pK_0{}^2(p) - pK_1{}^2(p)]/\pi, \tag{31}$$

Then w satisfies the following differential equation.

$$p^3 w''' + 2 p^2 w'' - (4p^2 + p)w' + w = 0 \tag{32}$$

We find

$$w_2 = \sum_0 a_n p^{-n}, \tag{33}$$

where the odd numbered coefficients are zero, $a_0 = 1$, $a_2 = -\frac{1}{8}$, and

$$a_n = (n^3 - 5n^2 + 7n - 3)/(4n a_{n-2}) \tag{34}$$

for n>2 and even.

We let $w_1 = e^{2p} w_1$, $w_3 = e^{-2p} w_3$. Then

$$p^3 w_1''' + (6p^3 + 2p^2)w_1'' + (8p^3 + 8p^2 - 2)w_1' + (8p^2 - 2p + 1) w_1 = 0 \tag{35}$$

with a similar equation for w_3. We find

$$W_1 = \sum_1 b_n p^{-n} \tag{36}$$

where $b_1 = 1$, $b_2 = 1/8$, and

$$(8n-8)\, b_n = (6n^2-14n+6)\, b_{n-1} - (n^3-5n^2+7n)b_{n-2}, \ n>2 \tag{37}$$

The function w_3 was treated in a similar manner. These formulas were programmed. We obtained 500 eigenvalues for a series of wall ratios ranging from .01 through 5, and the corresponding stresses at the outside radius, where the residue series is rapidly convergent.

V. STRESSES DUE TO AN ACCELERATING LOAD

We outline a method of analysis based on superposition, an eigenvalue expansion, interchange in the order of integration, and the evaluation of a complicated infinite integral. Justification for the various steps is omitted for brevity, but will be presented elsewhere in due course.

We assume the outside cylindrical surface is free of stress, but the inner boundary is subject to a discontinuous moving load.

$$\tau_{r\theta} = 0, \quad r = b \tag{39}$$

$$\tau_{r\theta} = \tau_0\, F_0(z,t) \tag{39}$$

where

$$F_0(z,t) = \tfrac{1}{2}, \quad z>T(t) \tag{40a}$$
$$= -\tfrac{1}{2}, \quad z<T(t) \tag{40b}$$

and $T(t)$ is the travel. We assume the velocity is subsonic, that is, $\dot{T}(t)<c_2$ where c_2 is the velocity of the shear wave in steel. In order to use separation of variables we assume

$$F_0(z,t) = \frac{1}{\pi} \int_0^\infty \frac{\sin[s\beta(z,t)]ds}{s} \tag{41}$$

where

$$\beta(z,t) = z - T(t) \tag{42}$$

We assume a solution of the form

$$\tau_{r\theta} = F_0(z,t) \left[R_0(r) - \sum_1 Q_n R(q_n,r) \right] + \sum_1 Q_n R(q_n,r) F_n(z,t) \tag{43}$$

where

$$R_0(r) = [a^2(b^4-r^4)] / [r^2(b^4-a^4)] \tag{44}$$

and

$$R(q_n,r) = A(q_n)[I(q_n r)K_2(q_n b) - K_2(q_n r)Y_2(q_n b)] \tag{45}$$

The eigenfunction expansion

$$R_0(r) = \sum_1 Q_n R(q_n,r) \tag{46}$$

can be obtained either by the theory of residues or the theory of orthogonal functions. We have used both methods to determine the Fourier coefficients Q_n and the results agree. The q_n are eigenvalues obtained from the characteristic equation $R(q,a) = 0$, and are purely imaginary since the problem is formulated in terms of modified Bessel functions.

We assume

$$F_n(z,t) = \frac{1}{\pi} \int_0^\infty \frac{W_n(s,z,t)}{s} \, ds \tag{47}$$

where

$$W_n(s,z,t) = G_n(s,t)\sin(sz) - H_n(st)\cos(sz) \tag{48}$$

On combining these results and substituting the value of $\tau_{r\theta}$ thus obtained in the differential equation

$$\frac{\partial^2 \tau_{r\theta}}{\partial r^2} + \frac{1}{\partial} \frac{\partial \tau_r}{\partial r^2} - \frac{4\tau_{r\theta}}{r^2} + \frac{\partial^2 \tau_{r\theta}}{\partial z^2} - \frac{1}{c_2^2} \frac{\partial^2 \tau_{r\theta}}{\partial t^2} \tag{49}$$

we obtain two ordinary linear inhomogeneous differential equations for a $G_n(s,t)$ and $H_n(s,t)$. We solve by Duhamels integral and evaluate $W_n(s,z,t)$. Duhamel's integral will appear inside the integral in Eq. (48). We interchange the order of integration. We obtain on letting $\alpha = t - t_1$,

$$F_n(z,t) = \int_0^t \int_0^\infty \frac{\sin[\alpha\sqrt{s^2 - q_n^2}]\sin\beta s \, ds \, dt_1}{c_2 \, s \sqrt{s^2 - q_n^2}} \tag{50}$$

where $\beta = z - T(t_1)$ in the above equation. We differentiate the inner integral partially with respect to β, evaluate the resulting inner integral by means of a known formula,* and integrate with respect to β to regain the original function $F_n(z,t)$. We obtain

$$F_n(z,t) = \tfrac{1}{2} Q_n \, c_2 \, p_n^2 \int_0^t \int_0^\alpha J_o \, p_n \sqrt{\alpha^2 - \beta^2(z,t_1)} \, d\beta dt_1 \tag{51}$$

Where $p_n^2 = -q_n^2$, and is real and positive. Thus we have two quadratures followed by a summation. In practice, the order summation and quadratures should be interchanged to reduce round off error.

*Reference 17, page 472, paragraph 3.876, Eq. (1)

We can readily obtain the response to a step function, then a square wave by superposition and translation. An additional convolution will account for variable torque, which must be obtained from the dynamics of the shell.

VI. DISCUSSION AND CONCLUSIONS

We have suggested methods of improving the accuracy of calculations based on classical analysis without using multiple precision calculations, and which are thus suitable for a group in engineering or applied mechanics. We have obtained formulas for stationary loads, loads moving with constant velocity, and, in the case of torsion, loads moving with arbitrary acceleration. The method presented here for solving the acceleration problem has not been found in the literature and therefore requires careful justification. Additional analysis and considerable programming are required to obtain codes for calculating the stresses and strains. Only then will the results be useful in interpreting strains obtained with instrumented gun tubes. The work is continuing with the time and resources available.

ACKNOWLEDGMENTS

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